9. Let Ω be a bounded open subset of $\mathbb{C}$, and $\varphi : \Omega \to \Omega$ a holomorphic function. Prove that if there exists a point $z_0 \in \Omega$ such that

$$\varphi(z_0) = z_0 \quad \text{and} \quad \varphi'(z_0) = 1$$

then φ is linear.

[Hint: Why can one assume that $z_0 = 0$? Write $\varphi(z) = z + a_n z^n + O(z^{n+1})$ near 0, and prove that if $\varphi_k = \varphi \circ \cdots \circ \varphi$ (where φ appears k times), then $\varphi_k(z) = z + k a_n z^n + O(z^{n+1})$. Apply the Cauchy inequalities and let $k \to \infty$ to conclude the proof. Here we use the standard O notation, where $f(z) = O(g(z))$ as $z \to 0$ means that $|f(z)| \leq C|g(z)|$ for some constant C as $|z| \to 0$.]

10. Weierstrass's theorem states that a continuous function on $[0, 1]$ can be uniformly approximated by polynomials. Can every continuous function on the closed unit disc be approximated uniformly by polynomials in the variable z?

11. Let f be a holomorphic function on the disc D_{R_0} centered at the origin and of radius R_0.

(a) Prove that whenever $0 < R < R_0$ and $|z| < R$, then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\varphi}) \text{Re}\left(\frac{Re^{i\varphi} + z}{Re^{i\varphi} - z}\right) d\varphi.$$

(b) Show that

$$\text{Re}\left(\frac{Re^{i\gamma} + r}{Re^{i\gamma} - r}\right) = \frac{R^2 - r^2}{R^2 - 2Rr\cos\gamma + r^2}.$$

[Hint: For the first part, note that if $w = R^2/\overline{z}$, then the integral of $f(\zeta)/(\zeta - w)$ around the circle of radius R centered at the origin is zero. Use this, together with the usual Cauchy integral formula, to deduce the desired identity.]

12. Let u be a real-valued function defined on the unit disc $\mathbb{D}$. Suppose that u is twice continuously differentiable and harmonic, that is,

$$\triangle u(x, y) = 0$$

for all $(x, y) \in \mathbb{D}$.

(a) Prove that there exists a holomorphic function f on the unit disc such that

$$\text{Re}(f) = u.$$

Also show that the imaginary part of f is uniquely defined up to an additive (real) constant. [Hint: From the previous chapter we would have $f'(z) = 2\partial u/\partial z$. Therefore, let $g(z) = 2\partial u/\partial z$ and prove that g is holomorphic. Why can one find F with $F' = g$? Prove that $\text{Re}(F)$ differs from u by a real constant.]

(b) Deduce from this result, and from Exercise 11, the Poisson integral representation formula from the Cauchy integral formula: If u is harmonic in the unit disc and continuous on its closure, then if $z = re^{i\theta}$ one has

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - \varphi)u(\varphi)\,d\varphi$$

where $P_r(\gamma)$ is the Poisson kernel for the unit disc given by

$$P_r(\gamma) = \frac{1-r^2}{1-2r\cos\gamma + r^2}.$$

13. Suppose f is an analytic function defined everywhere in $\mathbb{C}$ and such that for each $z_0 \in \mathbb{C}$ at least one coefficient in the expansion

$$f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$$

is equal to 0. Prove that f is a polynomial.
[Hint: Use the fact that $c_n n! = f^{(n)}(z_0)$ and use a countability argument.]

14. Suppose that f is holomorphic in an open set containing the closed unit disc, except for a pole at z_0 on the unit circle. Show that if

$$\sum_{n=0}^{\infty} a_n z^n$$

denotes the power series expansion of f in the open unit disc, then

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0.$$

15. Suppose f is a non-vanishing continuous function on $\overline{\mathbb{D}}$ that is holomorphic in $\mathbb{D}$. Prove that if

$$|f(z)| = 1 \qquad \text{whenever } |z| = 1,$$

then f is constant.
[Hint: Extend f to all of $\mathbb{C}$ by $f(z) = 1/\overline{f(1/\overline{z})}$ whenever $|z| > 1$, and argue as in the Schwarz reflection principle.]

7 Problems

1. Here are some examples of analytic functions on the unit disc that cannot be extended analytically past the unit circle. The following definition is needed. Let